F–Theory on Calabi-Yau Fourfolds

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Abstract

We discuss some aspects of F-theory in four dimensions on elliptically fibered Calabi-Yau fourfolds which are Calabi-Yau threefold fibrations. A particularly simple class of such manifolds emerges for fourfolds in which the generic Calabi-Yau threefold fiber is itself an elliptic fibration and is K3 fibered. Duality between F-theory compactified on Calabi-Yau fourfolds and heterotic strings on Calabi-Yau threefolds puts constraints on the cohomology of the fourfold. By computing the Hodge diamond of Calabi-Yau fourfolds we provide first numerical evidence for F-theory dualities in four dimensions.

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1. Introduction

F-theory has proven to be a useful framework for many of the string dualities [1, 2, 3, 4] which have been discussed in the last two years. This fact indicates that F-theory [5, 6] (together with M-theory [2, 7, 8]) might lead to a higher dimensional embedding of various types of string theories. The emphasis of recent papers [5, 6, 9] has been mostly on compactification of F-theory down to D=6 dimensions on Calabi-Yau threefolds[1].

In the present paper we discuss F-theory in four dimensions compactified on Calabi-Yau fourfolds. One of our tools is the generalization of the twist map of [11]. This map provides an explicit construction of K3-fibered Calabi-Yau threefolds by starting from a specific K3 surfaces with an automorphism and an associated higher genus Riemann surface. It shows to what extent the Heterotic/Type II duality in D=4 can be traced to string/string duality in D=6. Furthermore it isolates the additional structure of the fibration which is introduced by the twist of the fibration and is responsible for the dual type II image of the heterotic gauge structure. In Section 3 we generalize the twist map to construct $(n + 1)$-dimensional Calabi-Yau hypersurfaces which are fibered in terms of Calabi-Yau n-folds. We then use this map to derive and check consequences of F-theory duality in D=4.

Starting from Vafa's duality conjectures in D=6,8 we apply the twist map to Calabi-Yau threefolds to construct fourfolds whose generic fiber is the prescribed threefold. For F-theory one assumes that the Calabi-Yau space is an elliptic fibration. For such manifolds the expectation is that the twist map is concrete enough to allow for tests of the resulting D=4 conjecture relating[2]

$$F_{12}(CY_4) \longleftrightarrow Het(CY_3). \tag{1}$$

This conjecture thus provides a duality relation which involves a class of theories which are of phenomenological interest.

One way to generate a class of CY_3-fibered Calabi-Yau fourfolds which are also elliptically fibered is by considering CY_3-fibrations for which the generic fiber is itself a K3-fibered threefold for which the K3 in turn is elliptic. For such manifolds the gauge structure results of Heterotic/Type II duality predicts the dimension of cohomology groups of the Calabi-Yau fourfolds. This prediction can be tested. We take the first steps for such a check by computing the Hodge numbers for a variety of examples of Calabi-Yau fourfolds. The resolution structure of fourfolds is quite different from that of Calabi-Yau threefolds. We illustrate this difference by computing the cohomology of fourfolds for a number of different fibration types.

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[1]F-theory on K3×K3 has been considered in [10].

[2]We abbreviate Calabi-Yau n-folds by CY_n.

2. F-Theory Dualities in Various Dimensions

In this section we work our way down from F-theory in D=8 to D=4 dimensions.

2.1 F-Theory in D=8

It was argued in [5] that F-theory in D=8 compactified on an elliptic K3 surface is dual to the heterotic string on T^2

$$F_{12}(K3) \longleftrightarrow Het(T^2), \tag{2}$$

the fibration of K3 being described by $T^2 \longrightarrow K3 \longrightarrow \mathbb{P}_1$, where $\mathbb{P}_1$ denotes the base of the fibration whose typical fiber is the torus T^2. Compactifying F-theory further on a torus leads to

$$F_{12}(K3 \times T^2) \longleftrightarrow M_{11}(K3 \times S^1) \longleftrightarrow IIA(K3) \longleftrightarrow Het(T^4). \tag{3}$$

2.2 F-theory in D=6

In order to be able to push the D=8 duality of 2.1 down to D=6 by lifting it from K3 surfaces to Calabi-Yau threefolds one considers elliptically fibered CY_3s. A simple class of this type are threefolds which are K3 fibrations for which the generic fiber in turn is elliptically fibered. These spaces are simultaneous of the type $T^2 \longrightarrow CY_3 \longrightarrow B$ for some surface B, and of the type $K3 \longrightarrow CY_3 \longrightarrow \mathbb{P}_1$. For such manifold one might expect [5] to obtain the duality

$$F_{12}(CY_3) \longleftrightarrow Het(K3) \tag{4}$$

with the resulting chain of relations

$$F_{12}(CY_3 \times T^2) \longleftrightarrow M_{11}(CY_3 \times S^1) \longleftrightarrow IIA(CY_3) \longleftrightarrow Het(K3 \times T^2). \tag{5}$$

2.3 F-Theory in D=4

Our focus in the present paper is on compactifying F-theory down to four dimensions by considering Calabi-Yau fourfolds. To simplify the situation as much as possible we focus on fourfolds which are CY_3–fibered $CY_3 \longrightarrow CY_4 \longrightarrow \mathbb{P}_1$ such that the threefolds defining the generic smooth fiber are in turn elliptically fibered K3-fibration. For such manifolds it is natural to expect the duality

$$F_{12}(CY_4) \longleftrightarrow Het(CY_3) \tag{6}$$

and

$$F_{12}(CY_4 \times T^2) \longleftrightarrow M_{11}(CY_4 \times S^1) \longleftrightarrow IIA(CY_4) \longleftrightarrow Het(CY_3 \times T^2). \tag{7}$$

3. The Twist Map in Arbitrary Dimensions

3.1 The Twist Map

In order to understand the way lower dimensional dualities can be inherited from the higher dimensional ones, and in particular to see to what extent this is possible at all, it is useful to have a tool which constructs the necessary fibrations explicitly. Our way to do this employs the generalization of the orbifold construction of [11] to arbitrary dimensions. In the following we will call this generalized map the twist map. Our starting point is a Calabi-Yau n–fold with an automorphism group $\mathbb{Z}_\ell$ whose action we denote by $\mathbf{m}_\ell$. Furthermore we choose a curve $\mathcal{C}_g$ of genus $g = (\ell - 1)^2$ with projection $\pi_\ell : \mathcal{C}_g \longrightarrow \mathbb{P}_1$. The twist map then fibers Calabi-Yau n-folds into Calabi-Yau $(n + 1)$-folds

$$\mathcal{C}_g \times \mathrm{CY}_n \Big/ \mathbb{Z}_\ell \ni \pi_\ell \times \mathbf{m}_\ell \quad \longrightarrow \quad \mathrm{CY}_{n+1}. \tag{8}$$

For the class of weighted hypersurfaces

$$\mathbb{P}_{(k_0, k_1, ..., k_{n+1})}[k] \ni \{y_0^{k/k_0} + p(y_1, ..., y_{n+1}) = 0\}, \tag{9}$$

with $\ell = k/k_0 \in \mathbb{N}$ and $k = \sum_{i=0}^{n+1} k_i$, the cyclic action can be defined as

$$\mathbb{Z}_\ell \ni \mathbf{m}_\ell : \quad (y_0, y_1, ..., y_{n+1}) \longrightarrow (\alpha y_0, y_1, ..., y_{n+1}), \tag{10}$$

where α is the ℓth root of unity. An algebraic representation of the curve $\mathcal{C}_g$ is provided by

$$\mathbb{P}_{(2,1,1)}[2\ell] \ni \{x_0^\ell - \left(x_1^{2\ell} + x_2^{2\ell}\right) = 0\} \tag{11}$$

with action $x_0 \mapsto \alpha x_0$ and the remaining coordinates are invariant. The twist map in this weighted context takes the form

$$\mathbb{P}_{(2,1,1)}[2\ell] \times \mathbb{P}_{(k_0, k_1, ..., k_{n+1})}[k] \Big/ \mathbb{Z}_\ell \longrightarrow \mathbb{P}_{(k_0, k_0, 2k_1, ..., 2k_{n+1})}[2k] \tag{12}$$

and is defined as

$$((x_0, x_1, x_2), (y_0, y_1, ..., y_{n+1})) \longrightarrow \left(x_1 \sqrt{\frac{y_0}{x_0}}, x_2 \sqrt{\frac{y_0}{x_0}}, y_1, ..., y_{n+1}\right) \tag{13}$$

It is clear from the definition (13) that the twist map for hypersurfaces introduces additional singularites on the fibered $(n + 1)$–fold. In the simplest case, and the case of interest in the present context this additional singular set is the $\mathbb{Z}_2$ singular $(n - 1)$-fold $\mathbb{P}_{(k_1, ..., k_{n+1})}[k]$.

We will now focus on the special cases of $n = 2$ and $n = 3$ corresponding to the construction of weighted hypersurface fibrations of threefolds and fourfolds.

3.2 Construction of Fibered CY-Threefolds

For $n = 2$ one finds that the map introduces the $\mathbb{Z}_2$-singular curve $C = \mathbb{P}_{(k_1,k_2,k_3)}[k]$ which, on the threefold is in turn singular in general. When the resulting threefold fibrations are used in the context of Heterotic/Type II duality it is this additional curve which drastically changes the heterotic gauge structure one would expect if one were to focus solely on the K3 fiber. The reason for this is that in the process of pushing down the D=6 Heterotic/Type II duality to four dimensions

$$
\begin{array}{ccc}
\text{IIA(K3)} & \longleftrightarrow & \text{Het(T}^4) \\
\downarrow & & \downarrow \\
\text{IIA(CY}_3) & \longleftrightarrow & \text{Het(K3} \times \text{T}^2)
\end{array}
\tag{14}
$$

the twist introduces branchings of the Dynkin resolution diagram of the K3 surface by glueing together the various disconnected resolution diagrams of the surface. Thus it is this twist which in addition to the K3 singularity structure determines the gauge group.

Example I: As an example consider K3 Fermat type surface in $\mathbb{P}_{(1,2,6,9)}[18]$ with a $\mathbb{Z}_{18}$ automorphism, the associated curve being $\mathbb{P}_{(2,1,1)}[36]$. The resulting K3-fibered threefold $\mathbb{P}_{(1,1,4,12,18)}[36]$ has Hodge numbers $(h^{(1,1)}, h^{(2,1)}) = (7, 271)$. The heterotic gauge structure of this Calabi–Yau manifold is determined by the curve $C = \mathbb{P}_{(2,6,9)}[18]$ which glues together the three $\mathbb{Z}_4$–points, whose resolution lead to a total of 3 (1,1)–forms, and the $\mathbb{Z}_6$-point, whose resolution leads to 2 additional (1,1)-forms. Together with the Kähler form of the ambient space these modes provide $h^{(1,1)} = 7$. The intersection matrix of the resolution is precisely given by the Cartan matrix of the group SO(8)×U(1)^2.

Example II: We start with the K3 Fermat surface $\mathbb{P}_{(1,6,14,21)}[42]$. K has an automorphism group $\mathbb{Z}_{42}$ and we choose the curve as $\mathbb{P}_{(2,1,1)}[84]$. The image of the twist map is $\mathbb{P}_{(1,1,12,28,42)}[84]$. On the curve $C = \mathbb{P}_{(6,14,21)}[42]$ one finds a $\mathbb{Z}_2$, a $\mathbb{Z}_3$ and a $\mathbb{Z}_7$ fixed point, leading to 1, 2 and 6 new curves, respectively. Hence we have $h^{1,1} = 11$. The resolution diagram is given by E$_8$×U(1)^2 hence we see that the heterotic dual should be determined by Higgsing the first E$_8$ completely while retaining the second E$_8$. We also see that we should not fix the radii of the torus at some particular symmetric point but instead embed the full gauge bundle structure into the E$_8$.

Example III: Our final threefold is based on the K3 surface $\mathbb{P}_{(1,1,2,2)}[6]$ with a $\mathbb{Z}_6$ automorphism and the corresponding curve $\mathbb{P}_{(2,1,1)}[12]$. The resulting K3-fibration $\mathbb{P}_{(1,1,2,4,4)}[12]$ has Hodge numbers $(h^{(1,1)}, h^{(2,1)}) = (5, 101)$ and admits a conifold transition to a codimension two

Calabi–Yau manifold [13]. The heterotic gauge structure of this Calabi–Yau manifold is determined by the curve $C = \mathbb{P}_{(1,2,2)}[6] \sim \mathbb{P}_2[3]$ which glues together the three $\mathbb{Z}_2$–points whose resolution leads to a total of 3 (1,1)–forms. Together with the Kähler form of the ambient space we recover $h^{(1,1)} = 5$. The intersection matrix of the resolution is precisely given by the Cartan matrix of the group SO(8) and we see that in the heterotic dual we need to take the torus at the SU(3) point in the moduli space and break this SU(3) by embedding the K3 gauge bundle structure groups appropriately. More details for this manifold, first discussed in this context in [3], can be found in [11].

3.3 Construction of Fibered CY-Fourfolds

When pushing down the duality

$$
\begin{array}{ccc}
\text{IIA(CY}_3) & \longleftrightarrow & \text{Het(K3} \times \text{T}^2) \\
\downarrow & & \downarrow \\
\text{IIA(CY}_4) & \longleftrightarrow & \text{Het(CY}_3 \times \text{T}^2)
\end{array}
\tag{15}
$$

the singular curve on the generic Calabi-Yau fiber is embedded into the $\mathbb{Z}_2$-singular surface $\mathbb{P}_{(k_1,k_2,k_3,k_4)}[k]$. In particular for the two K3 fibrations discussed above the only singularities that appear on the resulting fourfolds lie on the $\mathbb{Z}_2$-singular curves.

Particularly simple classes of Calabi-Yau fourfolds, which are of interest in the context of duality, can be constructed by applying the twist map to the sequences of CY-threefolds discussed in [6] and [12]. These result in the classes of fourfolds

$$
\begin{array}{rcl}
\text{CY}_4^1(n) & := & \mathbb{P}_{(1,1,2,4n,8n+8,12n+12)}[24(n+1)] \\
\text{CY}_4^2(n) & := & \mathbb{P}_{(1,1,2,4n,4n+8,8n+12)}[8(2n+3))] \\
\text{CY}_4^3(n) & := & \mathbb{P}_{(1,1,2,4n,4n+8,4n+12)}[12(n+2)].
\end{array}
\tag{16}
$$

To be concrete consider the images of the twist map of the three examples of threefolds discussed in Section 3.2. The first two of these lead to fourfolds in the first sequence of (16) whereas the last example lives in neither of these classes. Now we know from the discussion above that the gauge group determined by the theory $\text{IIA}(\mathbb{P}_{(1,1,4,12,18)}[36])$ theory is SO(8) $\times \text{U}(1)^4$. Pushing down this theory to a fourdimensional F-theory via the twist map thus leads to the prediction that the second cohomology group of the fourfold $\mathbb{P}_{(1,1,2,8,24,36)}[72]$, since it measures the rank of the gauge group, should be 8-dimensional. Similarly the Calabi-Yau threefold $\mathbb{P}_{(1,1,12,28,42)}[84]$ leads to the prediction that the rank of the second cohomology group of the fourfold $\mathbb{P}_{(1,1,2,24,56,84)}[168]$ should be 12 whereas the $\text{IIA}(\mathbb{P}_{(1,1,2,4,4)}[12])$ theory leads to the expectation that the second cohomology group of the fourfold $\mathbb{P}_{(1,1,2,4,8,8)}[24]$ is 6-dimensional. It remains to compute the Hodge numbers of these spaces.

4. Calabi-Yau Fourfolds

In this section we check the predictions of the previous discussion. The examples we focus on for this purpose are mostly contained in the class defined by the first sequence of hypersurface fourfolds described in (16), which are fibered as

$$\mathbb{P}_{(1,1,n,2n+4,3n+6)}[6(n+2)] \quad \longrightarrow \quad \mathbb{P}_{(1,1,2,2n,4n+8,6n+12)}[12(n+2)] \\ \downarrow \\ \mathbb{P}_1 \qquad\qquad\qquad . \qquad (17)$$

Before coming to the computation of the Hodge number for such fibrations, however, it should be noted that the 'cohomology-behavior' of fourfolds is quite different from the behavior of threefolds. In contrast to threefold hypersurfaces, for which the simplest member, $\mathbb{P}_4[5]$ already leads to a representative cohomology Hodge diamond, this is not the case for Calabi-Yau fourfold hypersurfaces. The fourfold analog of the threefold quintic, the smooth sextic fourfold, already illustrates this: for $\mathbb{P}_5[6]$ the combined application of Lefshetz' hyperplane theorem, and counting complex deformations as well as computing the Euler number with the adjunction formula leads to the Hodge half-diamond

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
 1 & & 426 & & 1752 & & 426 & & 1
\end{array}
$$

resulting in the Euler number $\chi_4 = 2610$. The point here is that the third cohomology group vanishes $b_3 = 0$, a fact that does not hold for general 4folds as we will see below.

The methods just mentioned do not suffice for the computation of more general quasi-smooth Calabi-Yau fourfold hypersurfaces because one has to resolve the orbifold singularities. It turns out that the resolution of fourfolds is quite different from the resolution of three-folds. The resolution of both types of singularities, points [14] as well as curves [15] has been discussed in some detail for threefolds and differs markedly from the situation for fourfolds. Furthermore in Calabi-Yau fourfolds we encounter the situation where we have to resolve surfaces.

For any weighted Calabi-Yau fourfold one can use a combination of Cherning and resolu-

tion to compute the Euler number as

$$\chi_4 = \int_{\mathrm{CY}_4} c_4 - \sum_i \frac{\chi(S_i)}{n_i} + \sum_i n_i \chi(S_i), \tag{18}$$

where the S_i are the $\mathbb{Z}_{n_i}$ singular sets of the manifold.

For fourfolds which are CY$_3$-fibered there is a quicker and independent way to do this computation by using the fibration formula developed in [11]. Given a fibration whose generic smooth fiber is a Calabi-Yau threefold which degenerates over a finite number of points N_s of the base $\mathbb{P}_1$ into a cone over a surfaces $\mathcal{S}$ the Euler number follows. For the class of hypersurfaces (9) the formula becomes

$$\chi(\mathrm{CY}_4) = (2 - N_s)\chi(\mathrm{CY}_3) + N_s(\chi(\mathcal{S}) + k_0). \tag{19}$$

We postpone the detailed description of the geometric resolution to a more complete treatment and present here simply the results of our computations for the relevant manifolds.

We find for the Hodge half-diamond of the fourfold $\mathbb{P}_{(1,1,2,8,24,36)}[72]$, the image under the twist map of our first example in Section 3.2,

```
                          1
                  0               0
           0             8               0
      0          0             0             0
   1       6,528       26,188       6,528       1
```

with Euler number $\chi_4 = 39,264$. The latter is in agreement with the computation via the fibration formula (19). To see this it is sufficient to note that the generic smooth fiber degenerates over 72 points into the a surface $\mathcal{S}$ of Euler number $\chi(\mathcal{S}) = 31$. For the Hodge diamond of the second example $\mathbb{P}_{(1,1,2,24,56,84)}[168]$ we find

```
                          1
                  0               0
           0            12               0
      0          0             0             0
   1       27,548      110,284      27,548       1
```

with Euler number $\chi_4 = 165,408$. We may check this with Cherning. To do so we need first to enumerate the singularities of the hypersurfaces, leading to

$$\begin{aligned}
\mathbb{Z}_2 &: \quad S = \mathbb{P}_{(1,12,28,42)}[84] \\
\mathbb{Z}_4 &: \quad C = \mathbb{P}_{(6,14,21)}[42] \\
\mathbb{Z}_8 &: \quad \mathbb{P}_{(3,7)}[21] = 1 \; pt \\
\mathbb{Z}_{12} &: \quad \mathbb{P}_{(2,7)}[14] = 1 \; pt \\
\mathbb{Z}_{28} &: \quad \mathbb{P}_{(2,3)}[6] = 1 \; pt,
\end{aligned} \tag{20}$$

and compute the Euler numbers of the singular surface and the curve. With the fourth Chern class $c_4 = 222,223,000 \, h^4$ we then find

$$\begin{aligned}
\chi &= \frac{27,777,875}{168} - \frac{1}{2}\left(\frac{1091}{84} + \frac{1}{84}\right) + 2\left(\frac{1091}{84} + \frac{1}{84}\right) - \frac{1}{4}\left(-\frac{1}{42} - \frac{1}{2} - \frac{1}{3} - \frac{1}{7}\right) \\
&\quad + 4\left(-\frac{1}{42} - \frac{1}{2} - \frac{1}{3} - \frac{1}{7}\right) - \frac{1}{8} + 8 - \frac{1}{12} + 12 - \frac{1}{28} + 28 = 165,408.
\end{aligned} \tag{21}$$

in agreement with the Hodge diamond.

Finally, for the Hodge numbers of the third example $\mathbb{P}_{(1,1,2,24,56,84)}[168]$ we find

$$\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & 0 & & 0 & & & \\
& & 0 & & 6 & & 0 & & \\
& 0 & & 1 & & 1 & & 0 & \\
1 & & 803 & & 3{,}278 & & 803 & & 1
\end{array}$$

with Euler number $\chi_4 = 4,896$, which again can be checked with either the fibration formula or Cherning.

These numbers confirm the predictions of the analysis in the previous section. In Table 1 we list the results for a few other fourfolds. The emerging structure shows that the non-vanishing of b_3 is tied to the existence of singular sets of dimension one. It should be noted that the Euler number of all fibered manifolds considered here are divisible by 24, a necessary condition for anomaly cancellation[16].

No.	Manifold	χ	$h^{(1,1)}$	$h^{(2,1)}$	$h^{(3,1)}$	$h^{(2,2)}$	CY_3 Fibers, Gauge Group
1	$\mathbb{P}_5[6]$	2,610	1	0	426	1,752	–
2	$\mathbb{P}_{(1,1,2,2,2,2)}[10]$	2,160	2	0	350	1452	$\mathbb{P}_4[5]$
3	$\mathbb{P}_{(1,1,2,2,6,6)}[18]$	4,176	5	0	683	2,796	$\mathbb{P}_{(1,1,1,3,3)}[9]$
4	$\mathbb{P}_{(1,1,2,4,4,4)}[16]$	2,688	3	3	440	1,810	$\mathbb{P}_{(1,1,2,2,2)}[8]$
5	$\mathbb{P}_{(1,1,2,4,4,12)}[24]$	6,096	3	2	1,007	4,080	$\mathbb{P}_{(1,1,2,2,6)}[12]$, $U(1)^3$
6	$\mathbb{P}_{(1,1,2,4,8,8)}[24]$	4,896	6	1	803	3,278	$\mathbb{P}_{(1,1,2,4,4)}[12]$, $SO(8) \times U(1)^2$
7	$\mathbb{P}_{(1,1,2,4,16,24)}[48]$	23,328	4	1	3,876	15,566	$\mathbb{P}_{(1,1,2,8,12)}[24]$, $U(1)^4$
8	$\mathbb{P}_{(1,1,2,8,24,36)}[72]$	39,264	8	0	6,528	26,188	$\mathbb{P}_{(1,1,4,12,18)}[36]$, $SO(8) \times U(1)^4$
9	$\mathbb{P}_{(1,1,2,24,56,84)}[168]$	165,408	12	0	27,548	110,284	$\mathbb{P}_{(1,1,12,28,42)}[84]$, $E_8 \times U(1)^4$

Table *A short list for the cohomology of Calabi-Yau fourfolds of different fibration type. We also the record the known gauge groups.*

5. Conclusion

We have shown that pushing down the duality relations from $F_{12}(K3)$ to $F_{12}(CY_4)$

$$
\begin{array}{ccccccc}
F_{12}(K3 \times T^2) & \longleftrightarrow & M_{11}(K3 \times S^1) & \longleftrightarrow & IIA(K3) & \longleftrightarrow & Het(T^4) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{12}(CY_3 \times T^2) & \longleftrightarrow & M_{11}(CY_3 \times S^1) & \longleftrightarrow & IIA(CY_3) & \longleftrightarrow & Het(K3 \times T^2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{12}(CY_4 \times T^2) & \longleftrightarrow & M_{11}(CY_4 \times S^1) & \longleftrightarrow & IIA(CY_4) & \longleftrightarrow & Het(CY_3 \times T^2)
\end{array}
\tag{22}
$$

via the generalized twist map leads to predictions which can be confirmed.

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